

University of Groningen

Characters of the Nullcone

Hesselink, Wim H.

Published in:
Mathematische annalen

IMPORTANT NOTE: You are advised to consult the publisher's version (publisher's PDF) if you wish to cite from it. Please check the document version below.

Document Version
Publisher's PDF, also known as Version of record

Publication date:
1980

[Link to publication in University of Groningen/UMCG research database](#)

Citation for published version (APA):
Hesselink, W. H. (1980). Characters of the Nullcone. *Mathematische annalen*, 252(3), 179-182.

Copyright

Other than for strictly personal use, it is not permitted to download or to forward/distribute the text or part of it without the consent of the author(s) and/or copyright holder(s), unless the work is under an open content license (like Creative Commons).

The publication may also be distributed here under the terms of Article 25fa of the Dutch Copyright Act, indicated by the "Taverne" license. More information can be found on the University of Groningen website: <https://www.rug.nl/library/open-access/self-archiving-pure/taverne-amendment>.

Take-down policy

If you believe that this document breaches copyright please contact us providing details, and we will remove access to the work immediately and investigate your claim.

Downloaded from the University of Groningen/UMCG research database (Pure): <http://www.rug.nl/research/portal>. For technical reasons the number of authors shown on this cover page is limited to 10 maximum.

Characters of the Nullcone

Wim H. Hesselink

Mathematisch Instituut, Rijksuniversiteit Groningen, Postbus 800,
NL-9700 AV Groningen, Netherlands

1. Introduction

Let G be a semi-simple algebraic group over \mathbb{C} with Lie algebra \mathfrak{g} . The nullcone N of \mathfrak{g} consists of the nilpotent elements of \mathfrak{g} . Its co-ordinate ring $A(N)$ is a graded ring $\sum_{n \geq 0} A_n(N)$. We fix a maximal torus T of G and a dominant chamber C in the character group P of T . If $\lambda \in C$ let E_λ denote the irreducible G -module with highest weight λ . Let $d_n(\lambda)$ be the multiplicity of E_λ in the G -module $A_n(N)$.

Let R be the root system. Let R_+ be the set of the positive roots. Put $q := \frac{1}{2} \sum_{\alpha \in R_+} \alpha$. If $\chi \in P$ let $p_n(\chi)$ be the number of maps $f: R_+ \rightarrow \{0\} \cup \mathbb{N}$ such that $n = \sum_{\alpha \in R_+} f(\alpha)$ and $\chi = \sum_{\alpha \in R_+} f(\alpha)\alpha$. Let W be the Weyl group. If $w \in W$, put $R(w) := R_+ \cap -wR_+$ and $n(w) := \# R(w)$ and $\varepsilon(w) := (-1)^{n(w)}$. We prove the following theorem.

Theorem. If $\lambda \in C$ then $d_n(\lambda) = \sum_{w \in W} \varepsilon(w) p_n(w(\lambda + q) - q)$.

Remarks. The sequence $d_*(\lambda)$ is equivalent to the sequence of generalized exponents $m_*(\lambda)$ introduced by B. Kostant in [5]. In fact we have

$$d_n(\lambda) = \# \{i \mid m_i(\lambda) = n\} \quad \text{and} \quad m_i(\lambda) = \min \left\{ m \mid i \leq \sum_{n=0}^m d_n(\lambda) \right\}.$$

March 1976 I obtained the Theorem using the cohomological methods of [4]. After a conversation with T. A. Springer it became clear that the cohomology could be eliminated from the proof, see below. September 1978 I learned that D. Peterson had obtained the same Theorem, independently and by different methods.

2. We may assume that G is simply connected. The Grothendieck group $R(T)$ of the (finite dimensional) T -modules is identified with the group ring $\mathbb{Z}[P]$. If E is a T -module its class in $R(T)$ is denoted by $\text{ch}(E)$. Let V be an affine T -variety. Assume that the co-ordinate ring $A(V)$ is graded in such a way that the

homogeneous parts $A_n(V)$ are T -modules. Then we define the character of V to be the formal power series

$$\text{CH}(V, z) := \sum_{n=0}^{\infty} \text{ch}(A_n(V)) z^n.$$

Let E be a T -module with $\text{ch}(E) = \sum m(i) e^{\chi(i)}$, so $m(i) \in \mathbb{Z}$ and $\chi(i) \in P$. Since the coordinate ring of E is the symmetric algebra $S(E^*)$ on the dual E^* we obtain

$$\text{CH}(E, z) = \prod (1 - e^{-\chi(i)} z)^{-m(i)}.$$

3. Lemma. *If we put $W(z) := \sum_{w \in W} z^{n(w)}$ the nullcone N satisfies*

$$\text{CH}(N, z) = W(z) \prod_{\alpha \in R} (1 - e^{\alpha} z)^{-1}.$$

Proof. Put $r = \text{rank}(\mathfrak{g})$. By Sect. 2 we have

$$\text{CH}(\mathfrak{g}, z) = (1 - z)^{-r} \prod_{\alpha \in R} (1 - e^{\alpha} z)^{-1}.$$

B. Kostant has shown that $A(\mathfrak{g}) = H \otimes A(\mathfrak{g})^G$ where $H = \sum H_n$ and the G -module H_n is isomorphic to $A_n(N)$ for every n , cf. [5] Theorem 11. Let m_1, \dots, m_r be the exponents of the root system R . Put $d(i) = m_i + 1$. The ring of invariants $A(\mathfrak{g})^G$ is generated by algebraically independent homogeneous polynomials f_1, \dots, f_r of degrees $d(1), \dots, d(r)$. This implies

$$\sum \text{ch}(A_n(\mathfrak{g})^G) z^n = \prod (1 - z^{d(i)})^{-1}.$$

It follows that

$$\text{CH}(N, z) = (1 - z)^{-r} \prod_{i=1}^r (1 - z^{d(i)}) \prod_{\alpha \in R} (1 - e^{\alpha} z)^{-1}.$$

So the lemma follows from the identity

$$W(z) = (1 - z)^{-r} \prod_{i=1}^r (1 - z^{d(i)}), \text{ cf. [6] 2.6.}$$

4. The Weyl group action on P is extended to the ring $R(T)[[z]]$ in such a way that z is W -invariant. Let J be the endomorphism of $R(T)[[z]]$ given by $J = \sum \varepsilon(w) w$.

Lemma. *Let V be a subset of R_+ . Put $W(V) = \{w \in W \mid R(w) \subset V\}$. Then we have*

$$J \left(e^{\varrho} \prod_{\alpha \in V} (1 - e^{-\alpha} z) \right) = J(e^{\varrho}) \sum_{w \in W(V)} z^{n(w)}.$$

Proof. If $A \subset R_+$ we put $|A| = \sum_{\alpha \in A} \alpha$. We have

$$J \left(e^{\varrho} \prod_{\alpha \in V} (1 - e^{-\alpha} z) \right) = \sum_{A \subset V} (-z)^{|A|} J(e^{\varrho - |A|}).$$

In [6], p. 166, I.G. Macdonald proved that $J(e^{e^{-|A|}}) \neq 0$ if and only if $A = R(w)$ for some $w \in W$. Moreover the element w is necessarily unique. It satisfies $n(w) = \#A$ and $J(e^{e^{-|A|}}) = \varepsilon(w)J(e^e)$.

5. Proposition. $\text{CH}(N, z)J(e^e) = J\left(e^e \prod_{\alpha \in R_+} (1 - e^\alpha z)^{-1}\right)$.

Proof. Put $x = \prod_{\alpha \in R} (1 - e^\alpha z)^{-1}$ and $y = e^e \prod_{\alpha \in R_+} (1 - e^{-\alpha} z)$. Since x is W -invariant the righthand side of our formula is equal to $J(xy) = xJ(y)$. Now the equality follows from the Lemmas 3 and 4.

Remark. This proof is a simplification of an idea used in [4] p. 251.

6. Proof of the Theorem. The numbers $p_n(\chi)$ are determined by

$$\prod_{\alpha \in R_+} (1 - e^\alpha z)^{-1} = \sum_{\chi \in P, n \geq 0} p_n(\chi) e^\chi z^n.$$

The proposition implies

$$\begin{aligned} \text{ch}(A_n(N))J(e^e) &= \sum_{\chi \in P} p_n(\chi) J(e^{e+\chi}) \\ &= \sum_{\lambda \in C} \sum_{w \in W} \varepsilon(w) p_n(w(\lambda + \varrho) - \varrho) J(e^{\lambda + \varrho}). \end{aligned}$$

Since G -modules are characterized by their formal character, the Theorem follows by Weyl's character formula

$$\text{ch}(E_\lambda)J(e^e) = J(e^{\lambda + \varrho}),$$

cf. [1], Chap. 8, Sect. 9.

7. Remark. Let U be a maximal unipotent subgroup of G , normalized by T , whose weights are the positive roots. The ring of invariants $A(N)^U$ is a graded T -module with character

$$f(z) = \sum \text{ch}(A_n(N)^U) z^n = \sum_{\lambda \in C, n \geq 0} d_n(\lambda) e^\lambda z^n.$$

By the Theorem of Hadziev-Grosshans, cf. [3] and [2], the ring $A(N)^U$ is finitely generated. It follows that $f(z)$ is an element of the quotient field of the polynomial ring $R(T)[z]$.

Problem. Write $f(z)$ as a quotient $g(z)/h(z)$ with $g(z)$ and $h(z)$ in $R(T)[z]$ or rather in $\mathbb{Z}[C][z]$.

Remark. Let us define $D_n(\lambda) := \sum_{w \in W} \varepsilon(w) p_n(w(\lambda + \varrho) - \varrho)$ for all $\lambda \in P$, so that $D_n(\lambda) = d_n(\lambda)$ if $\lambda \in C$. The formal power series

$$F(z) := \sum_{\lambda \in P, n \geq 0} D_n(\lambda) e^\lambda z^n$$

satisfies

$$F(z) = e^{-\varrho} J\left(e^e \prod_{\alpha \in R_+} (1 - e^\alpha z)^{-1}\right).$$

Example 1. Let G be of type A_2 , with simple roots α and β . A weight $\lambda = x\alpha + y\beta$ satisfies $d_n(\lambda) = 1$ if and only if $d_n(\lambda) \neq 0$ if and only if x and y are integers with

$$\max\{2x - y, 2y - x\} \leq n \leq x + y.$$

It follows that

$$f(z) = \frac{1 + e^{\alpha + \beta} z^2 + e^{2\alpha + 2\beta} z^4}{(1 - e^{\alpha + \beta} z)(1 - e^{2\alpha + \beta} z^3)(1 - e^{\alpha + 2\beta} z^3)}.$$

Generators and relations for $A(N)^U$ are easily obtained.

Example 2. Let G be of type B_2 , with simple roots α (short) and β (long). A rather tedious calculation shows that

$$f(z) = \frac{1 + e^{3\alpha + 2\beta} z^4}{(1 - e^{\alpha + \beta} z^2)(1 - e^{2\alpha + 2\beta} z^2)(1 - e^{2\alpha + \beta} z)(1 - e^{2\alpha + \beta} z^3)}.$$

References

1. Bourbaki, N.: Groupes et algèbres de Lie, chap. 7 et 8. Paris: Hermann 1975
2. Grosshans, F.: Observable groups and Hilbert's fourteenth problem. Amer. J. Math. **95**, 229–253 (1973)
3. Hadziev, Dz.: Some questions in the theory of vector invariants. Math. USSR-Sbornik **1**, 383–396 (1967) (=Mat. Sbornik **72**, 114 No. 3 (1967))
4. Hesselink, W.H.: Cohomology and the resolution of the nilpotent variety. Math. Ann. **223**, 249–252 (1976)
5. Kostant, B.: Lie group representations on polynomial rings. Amer. J. Math. **85**, 327–404 (1963)
6. Macdonald, I.G.: The Poincaré series of a Coxeter group. Math. Ann. **199**, 161–174 (1972)

Received June 25, 1979